Exam Sets & Numbers (part 1: Sets),
February 2nd, 2024, 3:00pm–5:00pm,
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Write your name on every sheet of paper that you intend to hand in. Please provide **complete** *arguments for each of your answers. This part of the exam consists of* 2 *questions. You can score up to* 9 *points for each question, and you obtain* 2 *points for free. In this way you will score in total between* 2 *and* 20 *points.*

(1) (a) [2 points] Write down all subsets of the set $\{\pi, e\}$.
$\emptyset, \{\pi\}, \{e\}, \{\pi, e\}$.

(b) [2 points] Describe two subsets A, B of the set of natural numbers such that $\mathbb{N} \setminus A$ is countably infinite but $\mathbb{N} \setminus B$ is finite.
There are many options here given that $\mathbb{N}$ is countably infinite. Perhaps the simplest is $A = \emptyset$ and $B = \mathbb{N}$.

(c) [2+1 points] For any non-empty set X define $I = \{f \in X^X : f \text{ is injective }\}$ and $S = \{f \in X^X : f \text{ is surjective}\}$. Prove that $I \cap S$ is a non-empty set and that any element of $I \cap S$ must be invertible.
The identity id_X is both an injective and a surjective function from X to X so $\mathrm{id}_X \in I \cap S$. Therefore $I \cap S$ is not empty. Also injective and surjective is equivalent to being invertible by Theorem 4.2 of the lecture notes.

(d) [2 points] **Point out why the following is incorrect:** Suppose K, L, M are finite sets such that $K \subseteq M$ and $K \subseteq L$ and $\#L - \#M = 1$. Then

$$0 = \#L - \#K + \#K - \#L = \#(L \setminus K) + \#K - \#L = \#(K^c) + \#K - \#L$$

$$= \#(M \setminus K) + \#K - \#L = \#M - \#K + \#K - \#L = \#M - \#L = 1$$

There are in fact two mistakes here: an easy one is in the final equality: In fact $\#M - \#L = -1$. Another issue is that K^c is ambiguous: complement of K as subset of M or of L.

(2) Given two sets A and B recall the projection function $\pi_1 : A \times B \to A$ given by $\pi_1(a, b) = a$ for all $a, b \in A$. Also define for any $b \in B$ the function $\iota_b : A \to A \times B$ by $\iota_b(a) = (a, b)$ and the function $\sigma : A \times B \to B \times A$ by $\sigma(a, b) = (b, a)$

(a) [2 points] Prove that for all $b, b' \in B$ we have $\pi_1 \circ \iota_b = \pi_1 \circ \iota_{b'}$.
To show two functions are equal we need to show that they have the same domain and codomain and they give the same output for every element in the domain. For both functions the domain is A and the codomain is also A. To show that the functions are equal it remains to take any $a \in A$ and compute $(\pi_1 \circ \iota_b)(a) = \pi_1(\iota_b(a)) = \pi_1((a, b)) = a = \pi_1(\iota_{b'}(a)) = (\pi_1 \circ \iota_{b'})(a)$.

(b) [2 points] Assume $A = B$ and define a sequence of functions $f_0, f_1, f_2, \ldots$ by $f_0 = \mathrm{id}_{A^2}$ and $f_{n+1} = \sigma \circ f_n$. Prove by induction that for all $n \in \mathbb{N}$ we have $f_n \in \{\mathrm{id}_{A^2}, \sigma\}$.
We will prove the following more precise statement S_n for every $n \in \mathbb{N}$

using induction: If n is even the $f_n = \mathrm{id}_{A^2}$ and if n is odd then $f_n = \sigma$. It should be clear that S_n implies that $f_n \in \{\mathrm{id}_{A^2}, \sigma\}$. Induction basis: if $n = 0$ then n is even so this is consistent with $f_0 = \mathrm{id}_{A^2}$. Now assume that S_n is true for some n and then we will prove that S_{n+1} is also true. There are two options: either n is odd or n is even that have to be treated separately. If n is odd then by assumption S_n we have $f_n = \sigma$ so $f_{n+1} = \sigma \circ f_n = \sigma \circ \sigma = \mathrm{id}_{A^2}$, because for all $(a, a') \in A^2$ we have $\sigma \circ \sigma(a, a') = \sigma(a', a) = (a, a')$. Next if n is even then $f_n = \mathrm{id}_{A^2}$ so $f_{n+1} = \sigma \circ f_n = \sigma \circ \mathrm{id}_{A^2} = \sigma$ as required. This finishes the induction step an hence the proof.

(c) [2 points] Give an example of two specific sets A and B such that the function π_1 defined above is an invertible function.
When $A = B = \{1\}$ we can define a function $f : A \to A^2$ by $f(1) = (1, 1)$ and notice that $\pi_1 \circ f = \mathrm{id}_A$ because $\pi_1(f(1)) = \pi_1(1, 1) = 1$ while also $f \circ \pi_1 = \mathrm{id}_{A^2}$ because $f(\pi_1(1, 1)) = f(1) = (1, 1)$ showing that f is the inverse of π_1.

(d) [2+1 points] Suppose X is a finite set of sets and $Y \in X$. Define $R = \{(U, V) \in X^2 : U \cap Y = V \cap Y\}$. Prove that R defines an equivalence relation on X and describe the equivalence class of Y.
To check that R defines an equivalence relation we check the three properties reflexivity, symmetry and transitivity hold: Reflexivity: If $U \in X$ then $(U, U) \in R$ because $U \cap Y = U \cap Y$. Symmetry: If $(U, V) \in R$ then $U \cap Y = V \cap Y$ but then also $V \cap Y = U \cap Y$ so $(V, U) \in R$. Transitivity: If $(U, V) \in R$ and $(V, W) \in R$ then $U \cap Y = V \cap Y$ and $V \cap Y = W \cap Y$ so in particular $U \cap W$ showing that $(U, W) \in R$. Next, the equivalence class of Y consists of all $U \in X$ such that $U \cap Y = Y \cap Y = Y$. The sets such that $U \cap Y = Y$ are precisely the elements of X that contain Y as a subset.

If you are only retaking the sets part, this is the side you need to complete. Otherwise, please turn over for part 2 on numbers and do that part on a DIFFERENT piece of paper.